Tukey types of orthogonal ideals

Tomasz Żuchowski

Mathematical Institute, University of Wrocław

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Comparing posets $(P, \leq)$ and $(Q, \leq)$: Tukey reductions

Tukey reduction

We say that P is **Tukey reducible** to Q and write $P \preceq_T Q$ if there is a function $f : P \to Q$ such that $f^{-1}(B)$ is bounded in P whenever $B \subseteq Q$ is bounded. We call such function a **Tukey function** between P and Q.

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In other words...

$P \preceq_T Q$ means for every $q \in Q$ there is $h(q) \in P$ such that for every $x \in P$, if $f(x) \leq q$ then $x \leq h(q)$.
$h : Q \to P$ satisfies: $h(C)$ is cofinal in P for every cofinal $C \subseteq Q$.
Q is richer as a cofinal structure and $cf(P) \leq cf(Q)$.
Here $cf(Q)$ denotes the least cardinality of a set $C \subseteq Q$ which is **cofinal**, i.e. for every $q \in Q$ there is $c \in C$ with $q \leq c$.

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Notation

P and Q are **Tukey equivalent**, $P \equiv_T Q$, whenever $P \preceq_T Q$ and $Q \preceq_T P$.
$P \prec_T Q$ means $P \preceq_T Q$ but **not** $Q \preceq_T P$.

Families of subsets of ω

Orthogonal ideal

Given a family $\mathcal{A}$ of subsets of ω, its **orthogonal** is defined by $\mathcal{A}^\perp = \{B \subseteq \omega : A \cap B \text{ is finite for every } A \in \mathcal{A}\}$. It is always an ideal in $\mathcal{P}(\omega)$.

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Proof

If $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is adequate and $A \notin \mathcal{A}$ then there is finite $B \subseteq A$ such that $B \notin \mathcal{A}$ and we have $\{C \subseteq \omega : B \subseteq C\} \subseteq \mathcal{P}(\omega) \setminus \mathcal{A}$.

Five Tukey types

Five posets

We shall consider five posets: $\{0\}$, ω, ω^ω, $\mathcal{K}(\mathbb{Q})$ and $[\mathfrak{c}]^{<\omega}$.
They all are Tukey equivalent to poset $\mathcal{K}(X)$ of compact subsets of some topological space X ordered by inclusion:
$\{0\} \equiv_T \mathcal{K}([0,1])$, $\omega \equiv_T \mathcal{K}(\mathbb{N})$, $\omega^\omega \equiv_T \mathcal{K}(\mathbb{R}\setminus\mathbb{Q})$,
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Theorem (Aviles, Plebanek, Rodriguez)

Assume the axiom of analytic determinancy.
If $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is analytic as a subset of 2^ω then $\mathcal{A}^\perp$ is Tukey equivalent to one of posets $\{0\}$, ω, ω^ω, $\mathcal{K}(\mathbb{Q})$, $[\mathfrak{c}]^{<\omega}$.

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Sketch of the proof

From the results of Aviles, Plebanek and Rodriguez we have $\mathcal{K}_X(D) := \{B \subseteq D : cl(B) \in \mathcal{K}(X)\} \equiv_T \mathcal{K}(X)$. We will show that $\mathcal{K}_X(D) = \mathcal{A}^{\perp}$.

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Examples of adequate families

Corollary

There are $2^{\mathfrak{c}}$ different Tukey types of orders of form $\mathcal{A}^{\perp}$, where $\mathcal{A}$ is a family of subsets of ω.

Thus we have shown, that without any assumptions about a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we get as many Tukey types of $\mathcal{A}^{\perp}$ as there can be. Now we return to five posets defined earlier.

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Aviles has proved in abstract way, that for each of orders $[\mathfrak{c}]^{<\omega}$, $\mathcal{K}(\mathbb{Q})$, ω^{ω} and ω there exist an adequate family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathcal{A}^{\perp}$ is Tukey equivalent to that order. I have found concrete examples of adequate families with their orthogonals Tukey equivalent to that four orders.

Example of type $[\mathfrak{c}]^{<\omega}$

Remarks

- If P is a directed partial order of cardinality $\leq \mathfrak{c}$ then $P \preceq_T [\mathfrak{c}]^{<\omega}$. (Let $P = \{p_\xi : \xi < \mathfrak{c}\}$; then $p_\xi \to \xi$ is a Tukey reduction.)

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- To show $[\mathfrak{c}]^{<\omega} \preceq_T P$ we need to find $U = \{p_\xi : \xi < \mathfrak{c}\} \subseteq P$ such that no infinite subset of U is bounded.
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Family of branches in Cantor tree

Now for any $x \in 2^\omega$ we define $B(x) = \{x|n : n \in \mathbb{N}\}$ - a branch in Cantor tree $2^{<\omega}$ corresponding to x. The family $\mathcal{A} = \{A \subseteq B(x)$ for some $x \in 2^\omega\}$ is adequate in $\mathcal{P}(2^{<\omega})$.

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Combs

We define also **comb** $C(x)$ corresponding to $x \in 2^\omega$ as:
$C(x) = \{(x(0), 1 - x(1)),\ (x(0), x(1), 1 - x(2)),\ (x(0), x(1), x(2), 1 - x(3)),\ \ldots\}$
It's easy to see, that $C(x) \cap B(y)$ is a singleton for every $x, y \in 2^\omega$.

Example of type $[\mathfrak{c}]^{<\omega}$, continued

We have now a family $\{C(x) : x \in 2^\omega\}$ of size $\mathfrak{c}$ in $\mathcal{A}^\perp = \{B(x) : x \in 2^\omega\}^\perp$. We will show that this family meets the condition from the remark:

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and $x_0|m = (x_N(0), \ldots, x_N(m-1), 1 - x_N(m))$ is an element of $C(x_N)$. This shows, that $\bigcup\{C(x_n) : n \in \mathbb{N}\}$ contains infinitely many elements of $B(x_0)$.

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Corollary

$\{B(x) : x \in 2^\omega\}^\perp \equiv_T [\mathfrak{c}]^{<\omega}$

Thank you very much for your attention!!!